if p_0 is also constant everywhere. In other words, the flow has to be isentropic everywhere. Substituting Eq. (20a) into Eq. (21), it can be arranged as

$$\frac{\partial\, ln\{[1 + (\gamma - 1)M_a^2/2]/[1 + (\gamma - 1)M_a^2/4]\}}{\partial r} = -(\gamma - 1)(\omega^2 r/a_0^2)/[1 - (\gamma - 1)(\omega^2 r^2/2a_0^2)] \quad (22)$$

Direct integration of Eq. (22) also yields a closed form solution

$$\frac{[1 + (\gamma - 1)M_a^2/2]}{[1 + (\gamma - 1)M_a^2/4]} = C_2(z)[1 - (\gamma - 1)\omega^2 r^2/2a_0^2] \quad (23)$$

where $C_2(z)$ is an arbitrary function of z. $C_2(z)$ must be determined from the partial derivative of Crocco's equation with respect to z. For a constant area nozzle $C_2(z)$ is a constant which may be determined from the mass flow relationship.

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Wall Curvature and Transition Effects in Turbulent Boundary Layers

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Introduction

THIS Note presents one approach by which the eddy viscosity and mixing-length concepts which are being used in the current turbulent boundary-layer prediction methods can be modified to account for streamwise wall curvature and transition effects. A comparison of several calculated results using these modifications in the prediction method of [Ref. 1] show good agreement with experiment.

Analysis

We consider the momentum and energy equations for two-dimensional compressible boundary layers

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = u_e(du_e/dx) + (\partial \tau/\partial y) \quad (1)$$

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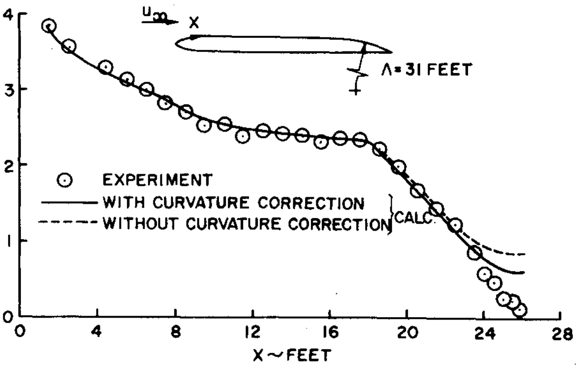

Fig. 1 Comparison of calculated experimental skin-friction coefficients for the data of Schubauer and Klebanoff.

$$\rho u(\partial H/\partial x) + \rho v(\partial H/\partial y) = (\partial/\partial y)(u\tau + q) \quad (2)$$

and the eddy viscosity formulation of Ref. 2, which for inner and outer regions of the boundary layer is defined by ϵ_i and ϵ_0, respectively

$$\epsilon = \begin{cases} \epsilon_i = (ky)^2(1 - \exp(-y/A))^2|\partial u/\partial y| \\ \epsilon_0 = 0.0168\left|\int_0^\infty (u_e - u)dy\right|\left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right]^{-1} \end{cases} \quad (3)$$

In Eq. (3), A is a damping-length constant given by

$$A = A^+(\tau_w/\rho_w)^{-1/2}(\rho/\rho_w)^{1/2}(1/N) \quad (4)$$

where

$$N \equiv \{(\mu/\mu_e)(\rho_e/\rho_w)^2\ p^+/v_w^+\ [1 - \exp(11.8\ (\mu_w/\mu)v_w^+)] + \exp(11.8\ (\mu_w/\mu)v_w^+\})^{1/4} \quad (5)$$

$$p^+ \equiv \frac{\nu}{u_e^2}\frac{du_e}{dx}\left(\frac{c_f}{2}\right)^{-3/2}, \quad v_w^+ \equiv v_w\left(\frac{\tau_w}{\rho_w}\right)^{-1/2} \quad (6)$$

The empirical constants k and A^+, which are generally assumed to be constants, vary at low Reynolds numbers[3]

$$k = 0.40 + 0.19/(1 + 0.49Z^2), \quad A^+ = 26 + [14/(1 + Z^2)], \quad Z \geq 0.3 \quad (7)$$

where $Z = R_\theta \times 10^{-3}$.

In Eqs. (1) and (2), τ and q denote the total shear stress and heat-transfer rate, respectively. They are given by

$$\tau = \mu(\partial u/\partial y) - \rho\langle u'v'\rangle, \quad q = k(\partial T/\partial y) - \rho\langle v'H'\rangle \quad (8)$$

which, by the eddy viscosity and turbulent Prandtl number (Pr_t) concepts, can also be written as (see Ref. 1)

$$\tau = \mu\frac{\partial u}{\partial y} + \rho\epsilon\frac{\partial u}{\partial y}, q = k\frac{\partial T}{\partial y} + \rho\frac{\epsilon}{Pr_t}\frac{\partial H}{\partial y} \quad (9)$$

The eddy viscosity expressions in the form given in Eq. (3) account for pressure gradient and heat and mass transfer effects quite well. They can be generalized to account for streamwise wall curvature effect by multiplying them (both inner and outer eddy viscosity expressions) by S^2, an ex-

Table 1 Extent of transitional Reynolds number at two blade Reynolds numbers

$Re_b \times 10^{-5}$	$Re_{tr} \times 10^{-5}$	$Re_{\Delta x} \times 10^{-5}$
1	0.1	0.28
1	0.5	0.81
10	1.0	1.30
10	5.0	3.80

pression given by Bradshaw.[4] This expression is based on an analogy between streamline curvature and buoyancy in turbulent shear flows. It is given by

$$S = 1/(1 + \beta R_i); \quad R_i = (2u/\Lambda)(\partial u/\partial y)^{-1} \tag{10}$$

The parameter β is equal to 7 for a convex surface and is equal to 4 for a concave surface. The radius of curvature Λ is positive for a convex surface and is negative for a concave surface. According to Bradshaw, the effects of curvature on the mixing length or eddy viscosity are appreciable if the ratio of boundary-layer thickness δ to radius of curvature Λ exceeds roughly 1/300.

In most of the practical boundary-layer calculations, it is necessary to calculate a complete boundary-layer field. That is, for a given pressure distribution, and for a given transition point (natural), it is necessary to calculate laminar, transition, and turbulent boundary layers by starting the calculations at the leading edge or at the stagnation point of the body. In current prediction methods, however, the calculation of transitional boundary layers are avoided by assuming this region to be just a switching *point* between laminar and turbulent boundary-layer calculations. At the transition point the turbulent boundary-layer calculations are started by activating the eddy-transport coefficients. In general, especially at low Reynolds numbers, this is not a good procedure and can lead to some errors. This point can best be described by an example. We consider the flow past a turbine or compressor blade and assume two blade Reynolds numbers Re_b of 10^5 and 10^6. The extent of the transition region on the blade at these two Reynolds numbers can be calculated by using the recent correlation given in Ref. 5

$$Re_{\Delta x} = A Re_{tr}^{2/3} \tag{11}$$

where $Re_{\Delta x}$ is the extent of the transition Reynolds number and A is an empirical expression given by

$$A = 60 + 4.86 M_e^{1.92}, \quad 0 < M_e < 5 \tag{12}$$

According to Ref. 5, Eqs. (11) and (12) are based on the correlation of incompressible and compressible data for Mach numbers less than 5. If we assume two transition points, namely, at 10 and 50% chord points, then the extent of the transition Reynolds number for the two blade Reynolds numbers (Re_b) according to Eq. (11) are shown in Table 1.

The previous values of $Re_{\Delta x}$ clearly show that the transitional region is very important and that it must be accounted

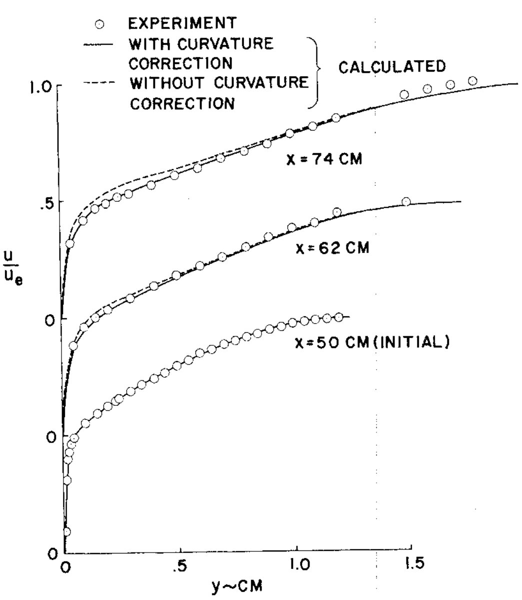

Fig. 2 Comparison of calculated and experimental velocity profiles for the data of Schmidbauer.

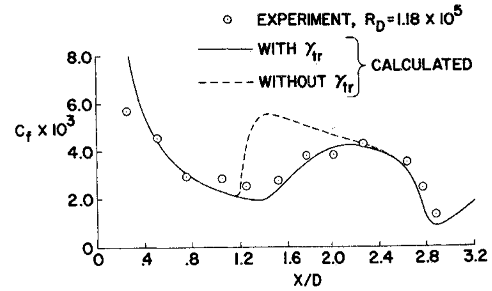

Fig. 3 Comparison of calculated and experimental skin-friction coefficients for Schubauer's ellipse.

for in order to make accurate boundary-layer calculations. For example, at a Reynolds number of 0.5×10^6, when the transition point is at 50% chord-point, the extent of transitional region is 81% which means that the flow in the rest of the body from the transition point up to the trailing edge is in a transitional state.

The eddy viscosity expressions given in Eq. (3) can also be generalized to account for the transition effects by multiplying them (expressions for both inner and outer regions) by the intermittency factor given by Chen and Thyson.[5] This expression is developed from the point of view of intermittent production of turbulent spots and is further extension of Emmons' spot theory[6] and Dhawan and Narasimha's intermittency expression for incompressible flows[7] to compressible flows with pressure gradient. According to Ref. 5, the intermittency factor is given by

$$\gamma_{tr} = 1 - \exp\left\{-Gr(x_{tr})\left[\int_{x_{tr}}^{x} \frac{dx}{r}\right]\left[\int_{x_{tr}}^{x} \frac{dx}{u_e}\right]\right\} \tag{13}$$

where G is a spot formation rate parameter,

$$G = (3/A^2)(u_e^3/\nu^2) Re_{tr}^{-1.34} \tag{14}$$

and

$$Re_{tr} = u_e x_{tr}/\nu \tag{15}$$

For compressible flows, the transition Reynolds number Re_{tr} can be satisfactorily calculated by using several empirical correlations. One such useful expression is the one based on combination of Michel's method[8] and Smith's e^9-correlation curve Ref. 9. It is given by (see Ref. 10)

$$R_{\theta_{tr}} = 1.174[1 + (22400/Re_{tr})]Re_{tr}^{0.46}, \quad 0.1 \leq 10^6 \leq Re_{tr} \leq 60 \times 10^6 \tag{16}$$

Comparison with Experiment

Figures 1–4 show a comparison of the calculated results with experiment. The calculations were made by using the method of Ref. 1, which for compressible flows consists of solving the system (1), (2), (9), and the continuity equation by using the eddy viscosity expressions given by Eq. (3) together with a constant turbulent Prandtl of 0.9 assumption. The method is applicable to both laminar and turbulent boundary layers. The calculations can be started either at the leading edge or at some downstream location. In the former case, the flow starts as laminar and becomes turbulent at any x location by activating the eddy-transport coefficients. In the latter case, it is necessary to specify the initial velocity profiles.

Figures 1 and 2 show the effect of wall curvature modification on the computed skin friction and velocity profiles for the experimental data of Schubauer and Klebanoff[11] and Schmidbauer[12], respectively. In the former case, δ/Λ is around 1/100, and in the latter case, δ/Λ is around 1/75. The wall-curvature correction definitely seems to improve the calculations.

Figures 3 and 4 show the effect of transition region modification on the computed skin friction for two different flows. Figure 3 shows the results for the experimental data

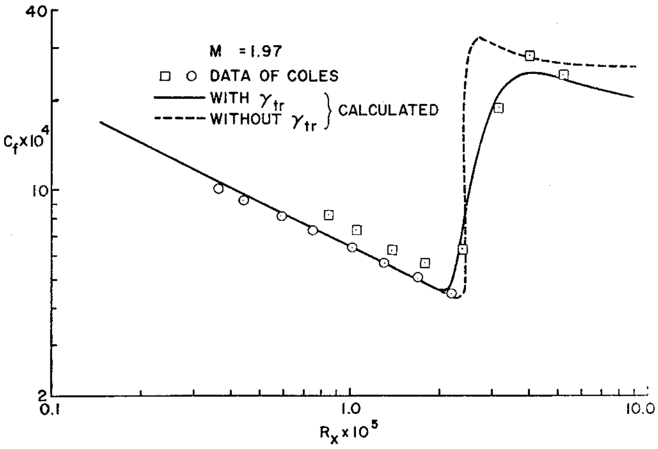

Fig. 4 Comparison of calculated and experimental skin-friction coefficients for a supersonic flat-plate flow measured by Coles.

of Schubauer.[13] This is an incompressible flow at relatively low Reynolds numbers. Figure 4 shows the results for the experimental data of Coles.[14] This is a supersonic adiabatic flow. The results indicate that in both cases the calculations by using the intermittency distribution given by Eq. (10) seem to account for the transition region rather well.

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Alleviation of Vortex-Induced Heating to the Lee Side of Slender Wings in Hypersonic Flow

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RECENT hypersonic studies show that the vortex system over the lee surface of planar delta wings,[1–3] blunted half-cones,[4,5] and on a conceptual high-cross range shuttle vehicle[6] can cause intense heating to the centerline region. The critical effect of Reynolds number in determining the location, initiation, and intensity of the vortex-induced heating has been studied.[4–6] Moore[7] indicates that vortices are produced at a leading edge corner, i.e., wherever a discontinuity exists in the leading-edge geometry. This observation led the present authors to propose contouring the planform of the apex region of a slender delta wing to more gradually turn the flow to reduce the interaction between opposing leading edge flows, which leads to the formation of the vortex system.

The present study was done in the Langley 11-in. Mach 6.8 Tunnel employing a sharp-apex delta wing, a rounded (circular-arc) apex delta wing, and hyperbolic and parabolic-planform wings. The leading edges of the models were sharp (<0.075 mm thick) with flat upper surfaces and lower surface bevel angles (perpendicular to the leading edge) between 18° and 20°. The sharp and rounded-apex delta wing models used in the oil-flow study were swept 75° so these results together with those reported in Refs. 1–3 provide information over at least a small range of sweep ($\Lambda = 70°, 75°, 78°$) for comparison with the hyperbolic and parbolic-planform wing results. The delta wings employed in the heating and vapor screen tests were swept 70°. Some details of the models are shown in Figs. 1 and 3. For the delta wings, the length L is the total length and for the hyperbolic and parabolic shape wings, L represents the distance along the root chord to the

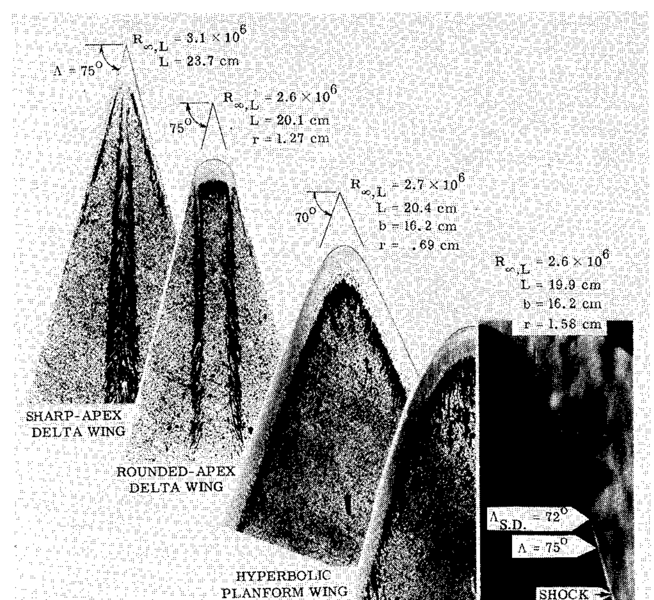

Fig. 1 Oil flow and Schlieren photographs; $\alpha = 7°$; $M = 6.8$.

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